Algebra-2

Final Examination

May 7, 2004

Instructions. All questions carry ten marks. All rings are assumed to be commutative with unity.

1. Define splitting field of a polynomial in $\mathbf{F}[X]$ where $\mathbf{F}$ is a field. Prove that for any $f \in \mathbf{F}[X]$, splitting field of f exists and is unique up to an isomorphism of fields.

2. Given n and distinct primes p and q, show that there is an isomorphism between the lattice of subfields of the field of order p^n and q^n.

3. State the Galois correspondence theorem, including the definitions of the relevant concepts. Find the Galois group of the splitting field of the polynomial $X^3 + X^2 - 3$ over $\mathbf{Q}$.

4. Define localisation of a ring. Let R be a ring. Prove that an ideal I of R is a prime ideal if and only if it is maximal with respect to the property of not meeting some multiplicatively closed subset of R.

5. Determine all the natural numbers n for which the angle of n degrees can be constructed by a ruler and a compass.

6. Let $K = \mathbf{Q}\left(e^{\frac{2i\pi}{n}}\right)$ and L be a cyclic (Galois) extension of degree n over K. Prove that there exists some $\alpha \in K$ such that $X^n - \alpha$ is an irreducible polynomial in $K[X]$ and $L = K(\alpha)$.

7. Let K be a field containing a primitive n-th root of unity for some natural number n. Suppose L is an n-Kummer extension over K. Use the description of cyclic extensions of degree n over K to show that $L = K(a_1, \ldots, a_r)$ with $a_i^n \in K$.

8. Let K be a field and let $f \in K[X_1, \ldots, X_n]$ be an irreducible polynomial. What is the transcendence degree of the function field of the variety of zeroes of f? Justify your answer.

9. Let $A \subset B$ be rings, B integral over A.

   (a) If $x \in A$ is a unit in B then prove that it is a unit in A.

   (b) Prove that the Jacobson radical of A is the contraction of the Jacobson radical of B.